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* Schrodinger's time dependent wave equation :-

Q. Derive Schrodinger's eqn. of time dependent & write the significance of wave function.

Sol: Inorder to derive time dependent Schrodinger wave eqn., Schrodinger introduced a mathematical function which associates characteristics of the Debroglie waves called as wave function denoted by ψ, which is a complex function.

The differential eqn representing one dimensional wave motion expressed mathematically,

$$\frac{\partial^2 \psi}{\partial t^2} = v^2 \cdot \frac{\partial^2 \psi}{\partial x^2}$$

in generally $\frac{\partial^2 \psi}{\partial t^2} = v^2 \nabla^2 \psi$ —— ①

the solution of eqn ① assumed to be in the x-direction by a wave function

$$\therefore \psi = A \cdot e^{-i\omega(t - \frac{x}{v})} \quad \text{———} \quad (2)$$

(or)

$$\psi = A \cdot e^{-i(\omega t - i\omega \frac{x}{v})}$$

where ~~ψ~~ ν = frequency $\omega = 2\pi\nu$ & $v = \nu\lambda$

$$\Rightarrow \psi = A \cdot e^{-i(2\pi\nu)\left[t - \frac{x}{\nu\lambda}\right]}$$

$$\psi = A \cdot e^{-2\pi\left[i\nu t - i\nu \cdot \frac{x}{\nu\lambda}\right]}$$

$$\psi = A \cdot e^{-2\pi i\left[\nu t - \frac{x}{\lambda}\right]} \quad \text{———} \quad (3)$$

Let E be total energy & P be the momentum of the particle then Quantum energy $E = h\nu$ and $\lambda = \frac{h}{P} \Rightarrow \nu = \frac{E}{h}$.

Substituting in eqn (2). we get

$$\psi = A \cdot e^{-2\pi i\left[\frac{E}{h} \cdot t - \frac{x \cdot P}{h}\right]}$$

$$\psi = A \cdot e^{-\frac{2\pi}{h}[Et - Px]} \qquad \left[\because \hbar = \frac{h}{2\pi} \quad \frac{2\pi}{h} = \frac{1}{\hbar}\right]$$

$$\Rightarrow \psi = A \cdot e^{-i/\hbar\,[Et - Px]} \quad \text{———} \quad (4)$$

diff. eqn (4) w.r.t "x"

$$\frac{\partial \psi}{\partial x} = A \cdot e^{-i/\hbar\,[Et - Px]} \cdot \left[\frac{i}{\hbar} \cdot P\right]$$

Again diff. w.r.t "x"

$$\frac{\partial^2 \psi}{\partial x^2} = A\left[\frac{i}{\hbar} \cdot P\right] \cdot e^{-i/\hbar\,(Et - Px)} \cdot \left(\frac{iP}{\hbar}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} = A\left[\frac{i^2 P^2}{\hbar^2}\right] \cdot e^{-i/\hbar\,(Et - Px)}.$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{(-1)\,P^2}{\hbar^2} \cdot A \cdot e^{-i/\hbar\,(Et - Px)}$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-P^2(4\pi^2)}{\hbar^2} \cdot A \cdot e^{-\frac{2\pi i}{\hbar}(Et - Px)}.$$

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{\hbar^2} P^2 \psi^2 \quad \text{———} (5).$$

At speeds the eqn(5) is the result of position of a particle interms of its momentum.

differentiating eqn(4) w.r.t 'time'. we get,

$$\frac{\partial \psi}{\partial t} = \frac{\partial}{\partial t}(e^{at})$$

$$= a \cdot e^{at}.$$

$$\frac{\partial \psi}{\partial t} = A \cdot e^{\frac{-2\pi i}{h}(Et - Px)} \left[-\frac{2\pi}{h} i E\right]$$

$$\frac{\partial \psi}{\partial t} = -\left[\frac{2\pi}{h} i E\right] \psi \quad \text{—— (6)}$$

The Speed of particle is very Small as Compared to Speed of light. then, the total energy E of particle is

$$E = K.E + P.E$$

$$E = \left(\frac{P^2}{2m} + V\right)$$

here P^2 & V are position & time.

multiplying ~~on~~ b.s on ψ

$$E\psi = \left[\frac{P^2}{2m} + V\right] \psi$$

$$E\psi = \frac{P^2}{2m} \psi + V \cdot \psi \quad \text{—— (7)}$$

Observing the eqns (6) and (5). we get from eqn (6).

$$\frac{\partial \psi}{\partial t} = \frac{-2\pi i}{h} E\psi$$

$$\frac{-h}{2\pi i} \cdot \frac{\partial \psi}{\partial t} = E\psi \quad \text{—— (8)}$$

$$i \cdot \frac{h}{2\pi} \cdot \frac{\partial \psi}{\partial t} = E\psi$$

(or)

$$i\hbar \frac{\partial \psi}{\partial t} = E\psi$$

from eqn (5) $\Rightarrow$ $$\frac{\partial^2 \psi}{\partial x^2} = -\frac{4\pi^2}{h^2} \cdot p^2 \psi$$

$$\frac{h^2}{4\pi^2} \cdot \frac{\partial^2 \psi}{\partial x^2} = -p^2 \psi$$

$$\frac{-h^2}{2m(4\pi^2)} \cdot \frac{\partial^2 \psi}{\partial x^2} = +\frac{p^2}{2m} \psi$$

$$\frac{-h^2}{8m\pi^2} \cdot \frac{\partial^2 \psi}{\partial x^2} = \frac{p^2}{2m} \psi \quad \text{——— (9)}$$

Substituting eqn (8) & (9) in eqn (7).

we get,

$$E\psi = \frac{p^2}{2m} \psi + V\psi$$

$$\frac{-h}{2\pi i} \cdot \frac{\partial \psi}{\partial t} = -\frac{h^2}{8\pi^2 m} \cdot \frac{\partial^2 \psi}{\partial x^2} + V\psi \quad \text{——— (10)}$$

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The above eqn is Called are Known as Schrodingers dependent wave eqn. in '+' x-direction.

⇒ In 3-dimensions

$$-\frac{h}{2\pi i}\cdot\frac{\partial\psi}{\partial t} = \frac{h^2}{8\pi^2 m}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+V\psi$$

$$-\frac{h}{2\pi i}\cdot\frac{\partial\psi}{\partial t} = \frac{h^2}{8\pi^2 m}\left[\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right]\psi+V\psi.$$

$$\boxed{-\frac{h}{2\pi i}\cdot\frac{\partial\psi}{\partial t} = \frac{h^2}{8\pi^2 m}\nabla^2\psi+V\psi.}$$

$$i^2 = -1.$$

$$\boxed{\frac{h\cdot i}{2\pi}\cdot\frac{\partial\psi}{\partial t} = \frac{h^2}{8\pi^2 m}\nabla^2\psi+V\psi}$$

$$\hbar = \frac{h}{2\pi}$$

$$\hbar^2 = \frac{h^2}{4\pi^2}$$

The above eqn. is Known as Schrodinger wave eqn.

* Schrodinger's time independent wave eqn. (or) steady state form of Schrodinger's wave equation :-

L.A.Q imp

Q. Derive an expression for Schrodinger's time independent wave equation and write the significance of wave function.

Sol : In many cases potential energy of a particle does not depends on time. Thus, potential is a function of position of a particle only. Hence, Schrodinger's time dependent wave eqn. may be simplified by removing all references to time.

⇒ From the one dimensional wave function expressed as $\psi = A \cdot e^{-\frac{2\pi}{h} i (Et - P \cdot x)}$.

along one direction.

$$\psi = A \cdot e^{-\frac{2\pi}{h} i Et} \cdot e^{-\frac{2\pi}{h} i (-Px)}$$

$$\psi = \underbrace{A \cdot e^{-\frac{2\pi i}{h}(Et)}}_{\psi_0} \cdot e^{\frac{2\pi i}{h}(Px)} \quad \text{—— (IA)}$$

$$\psi = A\, e^{2\pi i/h\,(Px)} \cdot e^{-\frac{2\pi i}{h}(Et)}$$

$$\psi = \psi_0 \cdot e^{-2\pi i/h\,(Et)}$$

⇒ In eqn (IB), ψ is the product of position dependent and time dependent.
differentiating (IB) w.r.t time (t).

$$\frac{\partial \psi}{\partial t} = -\frac{2\pi i E}{h} \cdot \psi_0 \cdot e^{-\left(\frac{2\pi i E}{h}\right)t} \quad \text{---} (2)$$

Again differentiating (IB) w.r.t "x" we get

$$\frac{\partial \psi}{\partial x} = e^{\frac{-2\pi i}{h}(Et)} \frac{\partial \psi_0}{\partial x}$$

$$\frac{\partial^2 \psi}{\partial x^2} = e^{\frac{-2\pi i}{h}(Et)} \cdot \frac{\partial^2 \psi_0}{\partial x^2} \quad \text{---} (3).$$

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Schrodinger time dependent wave eqn.

$$\left(\frac{hi}{2\pi}\right)\frac{\partial \psi}{\partial t} = \frac{-h^2}{8\pi^2 m} \cdot \frac{\partial \psi^2}{\partial x^2} + V\psi.$$

Sub the eqn (2) & (3) in the abv. eqn.
we get, $\left(\frac{hi}{2\pi}\right)\left(\frac{-2\pi i}{h} \cdot E\right)\psi = \frac{h^2}{8\pi^2 m}\left(e^{-\left(\frac{2\pi i}{h}\right)Et}\right) \frac{\partial^2 \psi}{\partial x^2} + \partial \psi.$

$$E\psi - V\psi = \frac{h^2}{8\pi^2 m}\left[\frac{\partial^2 \psi}{\partial x^2}\right]$$

$$\frac{8\pi^2 m}{h^2}(E - V)\psi = \frac{-\partial^2 \psi}{\partial x^2}$$

$$\boxed{\frac{\partial^2 \psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0} \quad \text{---} (4).$$

⇒ eqn (4) gives study form of Schrodinger's time independent wave eqn. along '+' x - direction of propogation.

⇒ For 3D Schrodinger time independent wave eqn.

$$\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} + \frac{\partial^2 \psi}{\partial z^2} + \frac{8\pi^2 m}{h}(E - V)\psi = 0.$$

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right]\psi + \frac{8\pi^2 m}{h}(E - V)\psi = 0$$

$$\boxed{\nabla^2 \psi + \frac{8\pi^2 m}{h^2}(E - V)\psi = 0}$$

The above eqn shows the Schrodinger time independent wave eqn.

Method 2 : Schrodinger time independent wave eqn :

⇒ Schrodinger eqn. is diff. eqn. for debraglie waves associates with the particle which describe its motion.

⇒ For the moving particle the Variable motion of a particle expressed mathematically by a function Called wave function denoted as 'ψ' for a steady state systems

(time independent)

1D eqn :

Here A = amplitude

λ = wavelength

diff. w.r.t "x" $\frac{\partial \psi}{\partial x} = A\left(\frac{2\pi}{\lambda}\right) \cos\left(\frac{2\pi x}{\lambda}\right)$

again diff w.r.t 'x'.

$$\frac{\partial^2 \psi}{\partial x^2} = -A \frac{4\pi^2}{\lambda^2} \cdot \sin\left(\frac{2\pi x}{\lambda}\right)$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{-4\pi^2}{\lambda^2} \psi \text{ ——— } (2)$$

⇒ The wavelength associated with the particle mass 'm' moving with velocity 'v' expressed from debroglie's waves.

$$\lambda = \frac{h}{mv}$$

$$\frac{1}{\lambda} = \frac{mv}{h}$$

$$\frac{1}{\lambda^2} = \frac{m^2 v^2}{h^2}$$

$$\frac{1}{\lambda^2} = \frac{2m\left(\frac{1}{2}mv^2\right)}{h^2} \text{ ——— } (3)$$

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⇒ For non relativistic approach $\frac{1}{2} mv^2$ is the K.E of the particle.

⇒ Thus K.E $= \frac{1}{2} mv^2 = (E - V)$

$(\because E = K.E + P.E = \frac{1}{2} mv^2 + V \Rightarrow E - V = \frac{1}{2}m$

Sub the above eqn. is eqn ③

$$\frac{1}{\lambda^2} = \frac{2m}{h^2}(E-V) \quad \text{—— (4)}$$

Using the value of $\frac{1}{\lambda^2}$ from eqn ② in eqn ④. we get, from eqn ②

$$-\left(\frac{1}{4\pi^2}\right)\frac{\partial^2\psi}{\partial x^2} = \frac{1}{\lambda^2}\psi.$$

$$\left[-\left(\frac{1}{4\pi^2}\right)\frac{\partial^2}{\partial x^2}\right] = \frac{2m}{h^2}(E-V)$$

$$\frac{-\partial^2}{\partial x^2} = \frac{8\pi^2}{h^2}(E-V).$$

$$-\frac{\partial^2\psi}{\partial x^2} = \frac{8\pi^2 m}{h^2}(E-V)\psi$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{8\pi^2 m}{h^2}(E-V)\psi = 0 \quad \text{—— (5)}$$

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⇒ The eqn ⑤ give time independent (steady state) Schrodinger eqn. for a particle of mass 'm', total energy 'E' and P.E 'V' moving along x-direction for a Space (3D) Schrodinger time independent wave eqn.

$$\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}\right)+\frac{8\pi^2 m}{h^2}(E-V)\psi=0$$

$$\boxed{\nabla^2\psi+\frac{8\pi^2 m}{h^2}(E-V)\psi=0.}$$

* Physical Significance of Wave Function :-

(or)

Properties of wave function :-

⇒ In 1926, Max Bohr gave a satisfactory interpretetion of a wave function. ψ associated a particle relation by defining wave packet.

⇒ wave packet :- A finite region existency of a particle gives its identity.

⇒ The Square of magnitude of

wave function also Called probability density which interprets the probability of finding a particle within a volume $dx\, dy\, dz$.

$$|\psi|^2 = \psi\psi^*$$

Mathematically $|\psi|^2 = \psi\psi^*$

$$\iiint |\psi|^2 dv = \iiint \psi\psi^* dx \cdot dy\, dz = 1.$$

This eqn. is Called probability finding function.

This Condition is Known as Normalized Condition.

⇒ Here the function ψ is also Called Amplitude function.

⇒ The wave function ψ must be finite everywhere. it voilets uncertainity principle (or zero value at any point).

⇒ ψ must be Single Value integration of

$$\int_a^b \psi_1^*(x)\, \psi_2(x) \cdot dx = 0.$$

This function is Called Orthogonal function.

⇒ $A\psi$ is a wave function multiplied by a Constant gives new function its Complex Conjugate $(A\psi)^*$ then its normalised function $\iiint (A\psi)^* (A\psi) \cdot dx\, dy \cdot dz = 1$.

$$\int (A\psi)^* (A\psi) \cdot dx = 1.$$

$$|A|^2 \int \psi^* \psi \cdot dx = 1.$$

$$A = \frac{1}{\int \psi^* \psi \cdot dx}.$$

In the above eqn. the function ψ or interms becomes Solutions of Schrodinger wave equations.

⇒ The function ψ must be Continuous across the boundary.

⇒ The function ψ always gives characteristics (or) properties of eigen Values (allowed energy states of a given function).

Mathematically expressed as $f(\psi)$.

* Application of Schrodinger ~~to~~ wave equation to the organic oscillator :-

L.A.Q

Q. Estimate the energy of Simple harmonic oscillator from Schrodinger wave mechanics.

(or)

Harmonic oscillator is a parabolic potential well. Support this phenomenon from Quantum wave mechanics.

(or)

Solve the linear harmonic oscillator problem quantum mechanically and obtain its eigen Values.

(or)

Establish Schrodinger's equation for a linear harmonic Oscillator and Solve it to obtain its eigen values and eigen functions.

Sol :- A simple harmonic oscillator is a particle executing simple harmonic motion along x-direction (1 D) under a restoring force 'F' acting for a unit displacement.

$$\therefore F = -Kx \quad \text{—— (4)}$$